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Scaling theory of antiferromagnetic Heisenberg ladder models

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Abstract. The $S = 1/2$ antiferromagnetic Heisenberg model on multi-leg ladders is investigated. Criticality of the ground-state transition is explored by means of finite-size scaling. The ladders with an even number of legs and those with an odd number of legs are distinguished clearly. In the former, the energy gap opens up as $\Delta E \sim J_{\perp}$, where J_{\perp} is the strength of the antiferromagnetic inter-chain coupling. In the latter, the critical phase with the central charge $c = 1$ extends over the whole region of $J_{\perp} > 0$.

1. Introduction

Understanding of the ground-state criticality of one-dimensional quantum system has been enriched greatly in the past decade. One of the important breakthroughs was finite-size scaling based on the conformal field theory [1]; see [2] for a review. Nowadays many attempts at generalization for higher-dimensional systems are in progress. One of them is the study of coupled chains. The Heisenberg ladder models are particularly of interest. The two-dimensional Heisenberg model might be explored as the limiting case of the ladder models. In addition, actual substances which realize the ladder models have been developed experimentally: $\text{Sr}_{2n-2}\text{Cu}_{2n}\text{O}_{4n-2}$, for example [3].

The Hamiltonian of the $S = 1/2$ Heisenberg ladder models is given by

$$\mathcal{H} \equiv \mathcal{H}_{\text{leg}} + \mathcal{H}_{\text{rung}} \quad (1.1)$$

where

$$\mathcal{H}_{\text{leg}} \equiv J \sum_{x=0}^{L-1} \sum_{y=1}^{n_l} \mathbf{S}_{x,y} \cdot \mathbf{S}_{x+1,y} \quad (1.2)$$

$$\mathcal{H}_{\text{rung}} \equiv J_{\perp} \sum_{x=0}^{L-1} \sum_{y=1}^{n_l-1} \mathbf{S}_{x,y} \cdot \mathbf{S}_{x,y+1}. \quad (1.3)$$

In the present paper we treat the antiferromagnetic case, $J > 0$ and $J_{\perp} > 0$. Each component of the spins is defined by

$$S^{\mu} = \frac{1}{2} \sigma^{\mu} \quad (\mu = x, y, z) \quad (1.4)$$

where $\{\sigma^{\mu}\}$ are the Pauli matrices. We define $S^{\pm} \equiv S^x \pm iS^y$. We impose periodic boundary conditions only in the x direction:

$$S_{L,y} = S_{0,y} \quad \text{for } y = 1, 2, \dots, n_l. \quad (1.5)$$

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Hereafter the system size L is even.

In the present paper, we focus on the ground-state criticality of the ladder models. It was conjectured [4] that the ladder models with even n_l and those with odd n_l behave quite differently from each other; for $J_\perp = 1$, the former is massive, while the latter is massless. Here we discuss the criticality for arbitrary positive J_\perp and for general n_l .

It is well known that the antiferromagnetic Heisenberg chain ($J_\perp/J = 0$) is massless [5]. Many questions arise when we introduce the inter-chain coupling $\mathcal{H}_{\text{rung}}$: for example, it has been quite controversial [6–9] whether the system with $n_l = 2$ becomes massive for any positive J_\perp or whether there exists a critical point at a finite value of J_\perp ; it has not been clarified whether the massless phase for odd n_l extends over the whole region of $J_\perp > 0$ or whether a massive phase exists in the region $0 < J_\perp < 1$; and the universality of the massless phase has not been discussed.

The purpose of the present paper is to answer the above questions on the basis of finite-size scaling theory. We introduce our scaling ansatz in section 2.1. The ansatz yields useful conclusions immediately. In particular, we conclude that the whole region of $J_\perp > 0$ is controlled by the stable fixed point at $J_\perp/J = \infty$. Thus we distinguish the systems with even n_l and those with odd n_l clearly: the former is massive for any J_\perp , while the latter is critical with the central charge $c = 1$ for any J_\perp . We present some results for even n_l and for odd n_l in sections 2.2 and 2.3, respectively. We particularly show the estimation of critical amplitudes. We give numerical confirmation of the scaling ansatz in section 2.1. In addition to that, we confirm the ansatz by means of perturbation theory in section 3.

2. Finite-size scaling of the energy gap

In this section, we introduce the finite-size scaling form of the energy gap. We numerically confirm the scaling ansatz. We present our conclusions drawn from the ansatz.

2.1. General arguments

The central scaling ansatz of the present paper is written in the form

$$\Delta E(L) \simeq \frac{J}{L} \tilde{\Delta}(x) = J_\perp \frac{\tilde{\Delta}(x)}{x} \quad \text{with } x \equiv \frac{LJ_\perp}{J} \quad (2.1)$$

where ΔE is the energy gap between the ground state and the first excited state, and $\tilde{\Delta}$ denotes the relevant scaling function. We show, in figure 1, numerical data obtained by the Lanczos method [10]. The data are scaled well over a finite region of x (namely $0 \leq x \leq 5$) for $n_l = 2, 4$, and over the whole region of x for $n_l = 3$. We confirm the scaling form (2.1) by means of perturbation theory in section 3.

The ansatz (2.1) implies that the inter-chain coupling is a relevant operator. The coupling $\mathcal{H}_{\text{rung}}$ drives the system away from the point $J_\perp/J = 0$, and makes the system renormalized to the limit $J_\perp/J \rightarrow \infty$ as $L \rightarrow \infty$; see figure 2. Thus the fixed point at $J_\perp/J = \infty$ controls the whole phase of the region $J_\perp > 0$. Apart from a correction to scaling, the thermodynamic limit $L \rightarrow \infty$ is equivalent to the limit $J_\perp/J \rightarrow \infty$.

We can thereby understand the difference between the systems with even n_l and those with odd n_l . For even n_l , the spins on each rung form an $S = 0$ singlet in the limit $J_\perp/J \rightarrow \infty$. The system at the fixed point $J_\perp/J = \infty$ is the $S = 0$ chain with the energy gap of the order of J_\perp [11]. For odd n_l , on the other hand, the spins on each rung form an $S = 1/2$ doublet in the limit $J_\perp/J \rightarrow \infty$. In the first-order perturbation of J , we have the same energy spectrum as of the $S = 1/2$ Heisenberg chain (see section 2.3 for details).

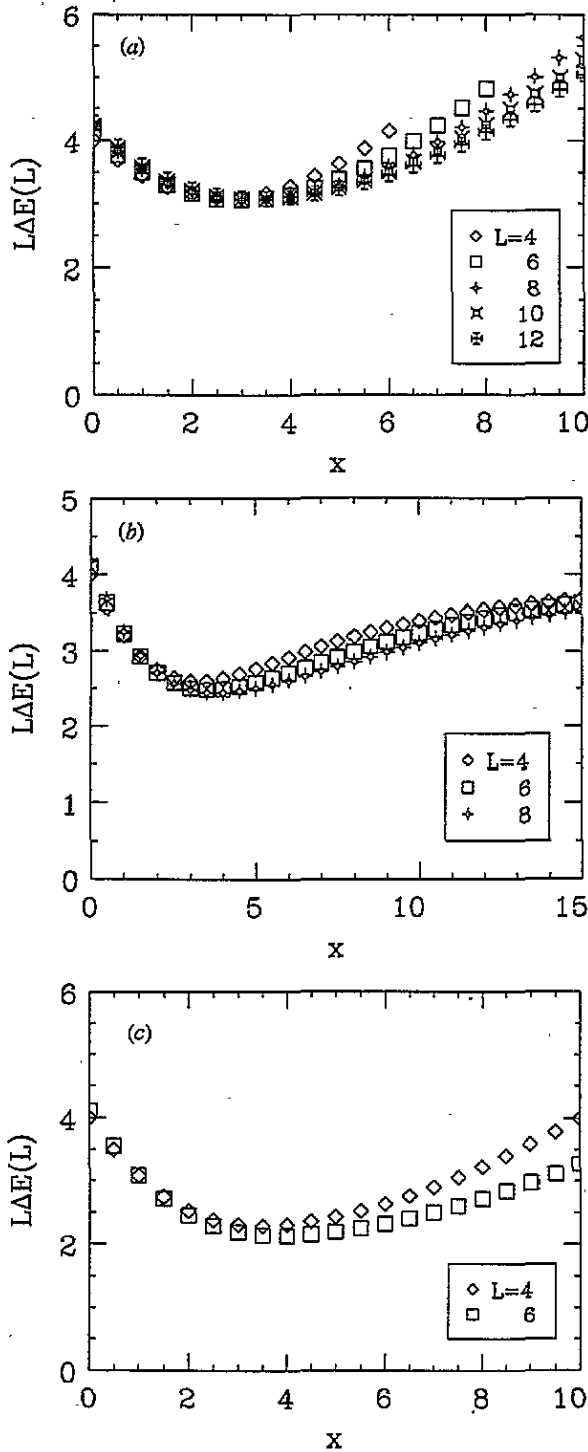


Figure 1. Scaling plot of numerical data for the J_{\perp} dependence of the energy gap. The abscissa is scaled as $x \equiv LJ_{\perp}/J$, while the ordinate is scaled as $L\Delta E/J$: (a) $n_l = 2$, $4 \leq L \leq 12$; (b) $n_l = 3$, $4 \leq L \leq 8$; (c) $n_l = 4$, $4 \leq L \leq 6$.

Hence the system at the fixed point $J_{\perp}/J = \infty$ is the $S = 1/2$ Heisenberg chain, which is massless, or critical. Thus the whole phase of $J_{\perp} > 0$ for even (odd) n_l is controlled by



Figure 2. The conjectured renormalization-flow diagram for the Heisenberg ladders. The fixed point $J_{\perp}/J = 0$ is unstable with respect to the inter-chain coupling. The stable fixed point at $J_{\perp}/J = \infty$ is described by the massive $S = 0$ chain for even n_l , and by the massless $S = 1/2$ chain for odd n_l . We thereby conclude that the whole phase of $J_{\perp} > 0$ is massive for even n_l and is critical for odd n_l .

the massive $S = 0$ (massless $S = 1/2$) Heisenberg chain.

In order to argue asymptotic behaviour for the scaling function $\tilde{\Delta}(x)$, we can exploit the two limits $J_{\perp}/J \rightarrow \infty$ and $L \rightarrow \infty$. In the thermodynamic limit $L \rightarrow \infty$, we must have a non-divergent value for the energy gap. Hence the scaling function $\tilde{\Delta}(x)$ in (2.1) must be of the first or lower order of x in the limit $x \rightarrow \infty$. We then postulate that the asymptotic form differs for even n_l and for odd n_l as follows:

$$\tilde{\Delta}(x) \sim \begin{cases} Ax & \text{for even } n_l \\ b & \text{for odd } n_l \end{cases} \quad \text{as } x \rightarrow \infty. \tag{2.2}$$

Here A and b are appropriate constants. We can derive this from the behaviour of the energy gap in the limit $J_{\perp}/J \rightarrow \infty$. For even n_l , a first excited state is created by exciting one of the $S = 0$ singlets on the rungs. The energy gap to this state is proportional to J_{\perp} : $\Delta E \simeq J_{\perp} \tilde{\Delta}(x)/x \propto J_{\perp}$. Therefore we should have $\tilde{\Delta}(x) \sim x$. For odd n_l , the energy gap for the first excited state is that of the $S = 1/2$ Heisenberg chain, and hence is proportional to J/L : $\Delta E \simeq (J/L) \tilde{\Delta}(x) \propto J/L$. This yields $\tilde{\Delta}(x) \sim \text{constant}$. The above postulate (2.2) is actually observed in the numerical data in figure 1.

2.2. Ladders with an even number of legs

According to the above ansatz (2.2), we have $\tilde{\Delta}(x)/x \rightarrow A$ as $x \rightarrow \infty$ for even n_l . Hence in the thermodynamic limit $L \rightarrow \infty$, the scaling form (2.1) is reduced to

$$\Delta E \simeq A J_{\perp}^{\nu} \quad \text{with } \nu = 1. \tag{2.3}$$

The coefficient A defined in (2.2) turns out to be the critical amplitude. Thus the critical system at $J_{\perp}/J = 0$ (namely, a set of independent Heisenberg chains) becomes massive immediately when we turn on the inter-chain coupling. The critical-point estimate $J_{\perp}/J = 0$ and the exponent estimate $\nu = 1$ were concluded previously [8, 9].

Let us estimate the critical amplitude A for $n_l = 2$. We utilize the following general finite-size scaling form for one-dimensional quantum ground-state phase transitions [12]:

$$\frac{L \Delta E(L)}{J} \simeq \begin{cases} Ax + D_+ e^{-Cx} & \text{in the disordered phase} \\ B & \text{at the critical point} \\ D_- e^{-Cx} & \text{in the ordered phase.} \end{cases} \tag{2.4}$$

Here $x \equiv L|\varepsilon|^{\nu}$ denotes the relevant scaling variable with ε being the distance from the critical point; $x = L J_{\perp}/J$ in the present case. The coefficient A gives the critical amplitude in (2.3). The coefficient C , on the other hand, gives the critical amplitude of the correlation length: $\xi^{-1} \simeq C(J_{\perp}/J)^{\nu}$. We introduced in [12] the amplitude relation

$$A/C = \nu_s \tag{2.5}$$

where v_s is the sound velocity. The coefficient B is predicted from conformal field theory as $B = v_s \pi \eta$, where η is Fisher's correlation exponent [13]. For the Hamiltonian (1.1) with $J_{\perp}/J = 0$, the Bethe-ansatz solution [14] gives

$$\eta = 1 \quad \text{and} \quad v_s = \frac{1}{2}\pi \quad (2.6)$$

hence we have

$$\frac{A}{C} = \frac{\pi}{2} = 1.570796\dots \quad \text{and} \quad B = \frac{\pi^2}{2} = 4.934802\dots \quad (2.7)$$

The coefficients D_+ and D_- may have weak x dependence [12].

The asymptotic behaviour in (2.2) for even n_l is consistent with the scaling form (2.4) in the disordered phase. We can employ our previous analysis in [12], where we showed for the one-dimensional quantum Potts model that a good estimator of the amplitude A is given by

$$A_L^{\min} \equiv \min_{x>0} \left(\frac{L\Delta E/J}{x} \right). \quad (2.8)$$

(If the data followed the scaling form (2.4) completely, the quantity $L\Delta E/(Jx)$ would converge to A exponentially as $x \rightarrow \infty$. However, there appears a minimum in practice, probably because of some correction to scaling [12].) The estimator (2.8) may converge to A exponentially [12] in the form

$$A_L^{\min} \simeq A + c_1 \exp(-c_2 L). \quad (2.9)$$

We calculated A_L^{\min} for the data in figure 1(a), and fitted the results to the form (2.9); see figure 3. We thereby have the estimate $A = 0.47(1)$ and hence $1/C = v_s/A = 3.34(7)$ for $n_l = 2$. (The error in the estimate A was evaluated through least-squares fitting to the form (2.9).) Namely, we have

$$\Delta E \simeq 0.47J_{\perp} \quad \text{and} \quad \xi \simeq 3.34 \frac{J}{J_{\perp}} \quad (2.10)$$

as $L \rightarrow \infty$ and near $J_{\perp}/J \sim 0$. These explain the following estimates in [4] quite well: $\Delta E = 0.504J$ and $\xi = 3.19(1)$ for $n_l = 2$ and $J_{\perp}/J = 1$. Although it is unnecessary that the critical behaviour (2.10) near $J_{\perp}/J \sim 0$ is observed even for $J_{\perp}/J = 1$, we see in figure 2 of [7] that the correction to (2.10) may be quite small in the region $J_{\perp}/J \leq 1$.

Incidentally, the energy gap is $\Delta E = J_{\perp}$ for $n_l = 2$ in the limit $J_{\perp}/J \rightarrow \infty$, where the system is reduced to a set of independent dimers. A crossover from the behaviour (2.10) to the behaviour $\Delta E = J_{\perp}$ may occur in the region $J_{\perp}/J > 1$.

If we naively apply the same analysis as above to the data for $n_l = 4$ with $L = 2, 4$ and 6, we obtain the tentative estimates

$$A \simeq 0.27 \quad \text{and} \quad C^{-1} = v_s/A \simeq 5.8. \quad (2.11)$$

Though there may be some errors in these estimates, the values (2.10) and (2.11) are fairly consistent with the scaling hypothesis $\xi \propto n_l$ [4].

2.3. Ladders with an odd number of legs

For odd n_l , we can see in figure 1 that the scaling region is quite wide. We may naturally assume that the scaling form (2.1) with (2.2) is valid for any value of L and J_{\perp}/J . Thus we have the following remarkable conclusion: for any positive J_{\perp} , the energy gap in the thermodynamic limit $L \rightarrow \infty$ behaves as

$$\frac{\Delta E}{J} \simeq \frac{b}{L} \quad (2.12)$$

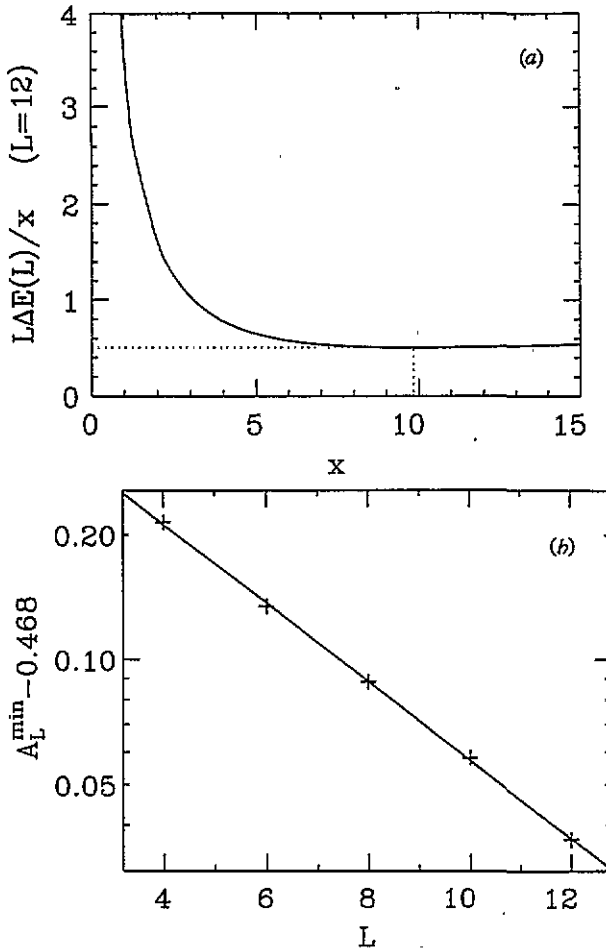


Figure 3. Estimation of the critical amplitude A for $n_l = 2$ as was done in [12]. (a) The quantity $L\Delta E/(Jx)$ against x for $L = 12$, for example. The estimator (2.8) is obtained as $A_{12}^{\min} = 0.505\dots$, which is indicated by the dotted lines. (b) The values of the estimator (2.8) are fitted to the form (2.9). The full line shows the best fit $A_L^{\min} \simeq 0.468 + 0.505 \exp(-0.218L)$.

where the amplitude b is the coefficient defined in (2.2). Note that this amplitude is independent of J_{\perp} by definition. In other words, the energy spectrum in the thermodynamic limit for any positive J_{\perp} is identical to that for $J_{\perp}/J \rightarrow \infty$, or the spectrum of the $S = 1/2$ antiferromagnetic Heisenberg chain. We thus conclude that the whole phase of $J_{\perp} > 0$ for odd n_l is a critical phase with the central charge $c = 1$. Note that the central charge is $c = n_l$ for $J_{\perp}/J = 0$, because we have n_l systems of $c = 1$ at this point.

We can obtain the value of the amplitude b by considering the limit $J_{\perp}/J \rightarrow \infty$. For this purpose, we first describe how to obtain explicitly the effective Hamiltonian of the $S = 1/2$ Heisenberg chain in the limit $J_{\perp}/J \rightarrow \infty$.

In the very limit of $J_{\perp}/J = \infty$, or $J/J_{\perp} = 0$, the Hamiltonian is reduced to $\mathcal{H}_{\text{rung}}$; the rungs are independent of each other. The ground state of the whole system is the direct product of the ground state of each rung. On each rung, an odd number of spins are coupled with the antiferromagnetic interaction J_{\perp} . Hence two ground states of each rung are degenerate with $S = 1/2$ and $S^z = \pm 1/2$. We express the states on the x th rung as $|+_x\rangle$ and $|-_x\rangle$. The ground states of the whole system have 2^L -fold degeneracy. This degeneracy is lift up in the first-order perturbation of J/J_{\perp} , or \mathcal{H}_{leg} . We calculate the

first-order perturbation for degenerate states by writing down the secular equation:

$$|\hat{H} - \lambda \hat{I}| = 0. \tag{2.13}$$

Here the matrix \hat{I} denotes the identity operator. The first-order energy is denoted by λ . The matrix \hat{H} is a $2^L \times 2^L$ matrix which represents the operator \mathcal{H}_{leg} in the subspace of the degenerate ground states. For example, one of the diagonal elements of \hat{H} is given by

$$\begin{aligned} & \langle -1 -2 +3 -4 \cdots | \mathcal{H}_{\text{leg}} | -1 -2 +3 -4 \cdots \rangle \\ &= J \sum_{x=0}^{L-1} \sum_{y=1}^{n_l} \langle -1 -2 +3 -4 \cdots | \mathcal{S}_{x,y} \cdot \mathcal{S}_{x+1,y} | -1 -2 +3 -4 \cdots \rangle \\ &= J \sum_{y=1}^{n_l} (\langle -1 | \mathcal{S}_{1,y}^z | -1 \rangle \langle -2 | \mathcal{S}_{2,y}^z | -2 \rangle + \langle -2 | \mathcal{S}_{2,y}^z | -2 \rangle \langle +3 | \mathcal{S}_{3,y}^z | +3 \rangle + \cdots) \end{aligned} \tag{2.14}$$

and one of the non-zero off-diagonal elements is given by

$$\langle -1 -2 +3 -4 \cdots | \mathcal{H}_{\text{leg}} | -1 +2 -3 -4 \cdots \rangle = \frac{J}{2} \sum_{y=1}^{n_l} \langle -2 | \mathcal{S}_{2,y}^- | +2 \rangle \langle +3 | \mathcal{S}_{3,y}^+ | -3 \rangle. \tag{2.15}$$

It is apparent that the secular equation (2.13) is equivalent to the eigenvalue equation for the $S = 1/2$ Heisenberg chain:

$$\mathcal{H}_{\text{eff}} = J_{\text{eff}} \sum_{x=0}^{L-1} \mathcal{S}_x \cdot \mathcal{S}_{x+1}. \tag{2.16}$$

We can see in (2.14) that the effective coupling J_{eff} is given by

$$J_{\text{eff}} = J \sum_{y=1}^{n_l} (\langle + | \mathcal{S}_y^z | + \rangle)^2. \tag{2.17}$$

We show numerical results for J_{eff} in figure 4. The value of J_{eff} is generally close to unity but increases monotonically. In particular, $J_{\text{eff}} = J$ for $n_l = 3$ [11]. For large n_l , the matrix element in (2.17) may depend on n_l as [2]

$$\langle + | \mathcal{S}_y^z | + \rangle \sim n_l^{-\beta/\nu} = n_l^{-1/2}. \tag{2.18}$$

Hence we may have $J_{\text{eff}} \sim O(n_l^0)$.

As we mentioned in (2.4) and (2.7), it is known [14] for the antiferromagnetic Heisenberg chain with the coupling J_{eff} that the energy gap behaves as

$$\Delta E \simeq \frac{v_s \pi \eta}{L} J_{\text{eff}} = \frac{\pi^2 J_{\text{eff}}}{2L}. \tag{2.19}$$

Comparing this with (2.12), we arrive at the conclusion

$$b \equiv \lim_{x \rightarrow \infty} \tilde{\Delta}(x) = v_s \pi \eta \frac{J_{\text{eff}}}{J} = \frac{\pi^2 J_{\text{eff}}}{2J}. \tag{2.20}$$

3. Perturbational derivation of the scaling ansatz

In this section, we describe a perturbational calculation which yields the scaling form (2.1). This was briefly reported in [8].

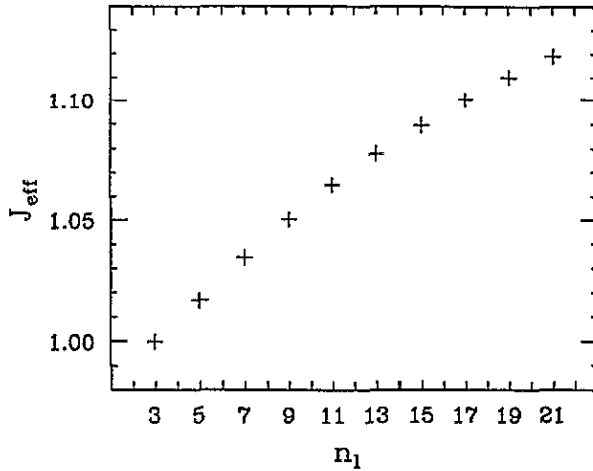


Figure 4. The effective coupling J_{eff} for odd n_l in the limit $J_{\perp}/J \rightarrow \infty$. The values were calculated numerically on the basis of (2.17).

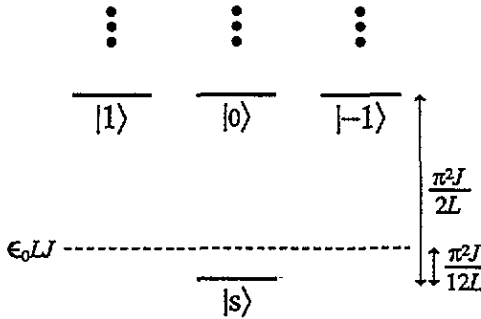


Figure 5. A schematic view of the low-lying energy spectrum of the $S = 1/2$ antiferromagnetic Heisenberg chain.

3.1. Zeroth order of J_{\perp}/J

Let us first describe the energy spectrum for $J_{\perp}/J = 0$. At this point, we have n_l number of the $S = 1/2$ Heisenberg chains independent of each other. The spectrum of each chain is given [14] as figure 5. Let $|s_y\rangle$ denote the singlet ground state of the y th leg, and let $|M_y\rangle = \{|1_y\rangle, |0_y\rangle, |-1_y\rangle\}$ denote the triplet of the first excited states. The energy of the singlet ground state is written in the form

$$E_s \simeq \epsilon_0 L J - \frac{v_s \pi c}{6L} J = \left(\epsilon_0 L - \frac{\pi^2}{12L} \right) J \quad \text{as } L \rightarrow \infty \quad (3.1)$$

where $\epsilon_0 = 1/4 - \ln 2$ is the exact ground-state energy density [5], $v_s = \pi/2$ is the sound velocity as before, and $c = 1$ denotes the central charge. The energy gap to the triplet states is given by

$$\Delta E \simeq \frac{v_s \pi \eta}{L} J = \frac{\pi^2}{2L} J \quad \text{as } L \rightarrow \infty \quad (3.2)$$

where $\eta = 1$ is Fisher's correlation exponent as before.

The ground state of the whole system is given by the direct product of the singlet states as

$$|\psi_{gs}^{(0)}\rangle = \bigotimes_{y=1}^{n_l} |s_y\rangle \tag{3.3}$$

with the energy

$$E_{gs} \simeq n_l \left(\epsilon_0 L - \frac{\pi^2}{12L} \right) J. \tag{3.4}$$

The first excited states have the $3n_l$ -fold degeneracy:

$$|\psi_{ex}^{(0)}\rangle = |M_{y_1}\rangle \otimes \left(\bigotimes_{\substack{y=1 \\ y \neq y_1}}^{n_l} |s_y\rangle \right) \quad (M = 0, \pm 1 \text{ and } y_1 = 1, 2, \dots, n_l). \tag{3.5}$$

The energy gap to these excited states is the same as in (3.2).

3.2. First-order perturbation of J_{\perp}/J

Now we calculate the first-order perturbation with respect to \mathcal{H}_{rung} . The first-order energy of the ground state is given by

$$\begin{aligned} E_{gs}^{(1)} &= \langle \psi_{gs}^{(0)} | \mathcal{H}_{rung} | \psi_{gs}^{(0)} \rangle \\ &= J_{\perp} \sum_{x=0}^{L-1} \sum_{y=1}^{n_l-1} \langle s_y | S_{x,y} | s_y \rangle \cdot \langle s_{y+1} | S_{x,y+1} | s_{y+1} \rangle \\ &= 0 \end{aligned} \tag{3.6}$$

because we have

$$\langle s | S_x^z | s \rangle = \langle s | S_x^{\pm} | s \rangle = 0. \tag{3.7}$$

In order to obtain the first-order energy of the excited state, we have to construct the secular equation, because the zeroth-order first excited states are degenerate. We concentrate on the sector $\sum_y M_y = 1$. The ground state of this sector gives the perturbed first excited state, because the perturbed energy spectrum would be similar to that in figure 5 owing to the SU(2) symmetry. The following n_l states are of this sector:

$$|(y_1)\rangle \equiv |1_{y_1}\rangle \otimes \left(\bigotimes_{\substack{y=1 \\ y \neq y_1}}^{n_l} |s_y\rangle \right) \quad (y_1 = 1, 2, \dots, n_l). \tag{3.8}$$

We immediately have

$$\langle (y_1) | \mathcal{H}_{rung} | (y_2) \rangle = 0 \quad \text{unless } |y_1 - y_2| = 1. \tag{3.9}$$

The non-zero matrix elements are

$$\begin{aligned} \langle (y+1) | \mathcal{H}_{rung} | (y) \rangle &= \frac{J_{\perp}}{2} \sum_{x=0}^{L-1} \langle s_y | S_{x,y}^- | 1_y \rangle \langle 1_{y+1} | S_{x,y+1}^+ | s_{y+1} \rangle \\ &= J_{\perp} a \quad (y = 1, 2, \dots, n_l - 1) \end{aligned} \tag{3.10}$$

and their conjugates. Here the coefficient a is defined by

$$a \equiv \frac{1}{2} L |\langle s | S_x^- | 1 \rangle|^2 \tag{3.11}$$

(The x dependence of the matrix element $\langle s|S_x^-|1\rangle$ appears only in the phase factor in the form e^{ikx} , and hence the absolute value is independent of x .) The secular equation for the first-order energy of the excited state is thereby written in the form

$$\begin{vmatrix} -E_{\text{ex}}^{(1)} & J_{\perp}a & & & 0 \\ J_{\perp}a & -E_{\text{ex}}^{(1)} & J_{\perp}a & & \\ & & J_{\perp}a & \ddots & \\ & & & \ddots & J_{\perp}a \\ 0 & & & & J_{\perp}a & -E_{\text{ex}}^{(1)} \end{vmatrix} = 0. \quad (3.12)$$

This is equivalent to the one-dimensional tight-binding model under the free boundary condition; the triplet state hops from a ladder to a neighbouring ladder with the hopping amplitude $J_{\perp}a$. This model is exactly solvable [15]. The zeroth-order wavefunction and the first-order energy are obtained in the forms

$$|\psi_{\text{ex}}^{(0)}\rangle = \sqrt{\frac{2}{n_l}} \sum_{y=1}^{n_l} \sin\left(\frac{m\pi}{n_l+1}y\right) |y\rangle \quad (3.13)$$

$$E_{\text{ex}}^{(1)} = -2J_{\perp}a \cos \frac{m\pi}{n_l+1} \quad (3.14)$$

for $m = 1, 2, \dots, n_l$. The state with $m = 1$ gives the lowest energy.

We thereby obtain the energy gap up to the first order of J_{\perp}/J as

$$\Delta E \simeq \frac{J}{L} \left(\frac{\pi^2}{2} - 2xa \cos \frac{\pi}{n_l+1} \right) \quad (3.15)$$

where $x \equiv LJ_{\perp}/J$ as before.

We are in position to estimate the size dependence of the coefficient a . In the following we show analytically and numerically that $a = O(L^0)$.

We first show that the coefficient a is proportional to the transverse susceptibility of the Heisenberg chain. Let us consider the Hamiltonian with a transverse field at the origin:

$$\bar{\Delta}e\mathcal{H} \equiv J \sum_{i=0}^{L-1} S_i \cdot S_{i+1} - \gamma S_0^x. \quad (3.16)$$

After the standard calculation, we have

$$\begin{aligned} \chi_{00} &\equiv \frac{\partial}{\partial \gamma} \langle S_0^x \rangle_{\gamma} \Big|_{\gamma=0} \\ &= \sum_{\psi_n (\neq \psi_{\text{gs}})} \frac{|\langle \psi_n | S_0^x | \psi_{\text{gs}} \rangle|^2}{\Delta E_n} \end{aligned} \quad (3.17)$$

$$= \int_0^{\infty} d\tau \langle S_0^x S_0^x(\tau) \rangle_0 \quad (3.18)$$

where the angular brackets $\langle \dots \rangle_{\gamma}$ denote the expectation value with respect to the ground state of (3.16), ΔE_n is the excitation energy of the excited state $|\psi_n\rangle$, and

$$S_0^x(\tau) \equiv e^{-\tau \bar{\Delta}e\mathcal{H}} S_0^x e^{\tau \bar{\Delta}e\mathcal{H}}. \quad (3.19)$$

The most dominant of the excited states $|\psi_n\rangle$ is $|S = 1, S^z = \pm 1\rangle$ with the energy gap (3.2), or $\Delta E \sim L^{-1}$. Hence the expression (3.17) is approximately rewritten as

$$\chi_{00} \simeq \frac{4a}{\pi^2}. \quad (3.20)$$

On the other hand, a field-theoretic description [16] yields

$$\langle S_0^x S_0^x(\tau) \rangle_0 \sim \frac{1}{\tau}. \tag{3.21}$$

Thus the expression (3.18) is a dimensionless quantity: $\chi_{00} = O(L^0)$. We thus arrive at the conclusion $a = O(L^0)$. There may be logarithmic corrections to this size dependence.

We also calculated the coefficient a in (3.11) numerically for $L \leq 24$; see figure 6. The logarithmic plot reveals that the coefficient actually behaves as

$$a \sim (\ln L)^{0.68} = O(L^0). \tag{3.22}$$

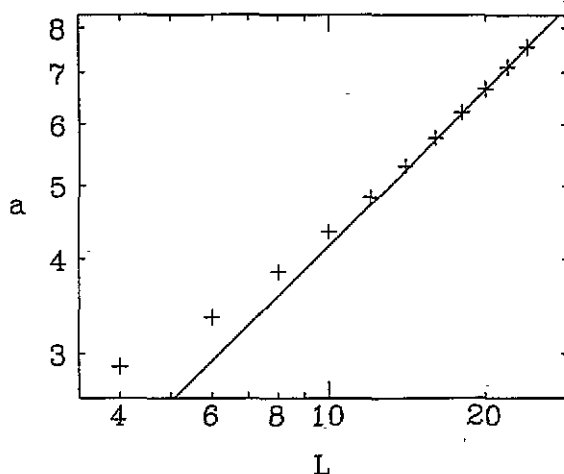


Figure 6. The size dependence of the quantity a appears to be of the form $a \sim (\ln L)^\omega$. The full line connects the last two data points with the slope $\omega \simeq 0.68$.

3.3. Order estimation of higher-order perturbations

We next derive the approximate second-order energy of the ground state. The second-order perturbation is given by the formula

$$E_{gs}^{(2)} = - \sum_{\psi_n (\neq \psi_{gs})} \frac{|\langle \psi_n | \mathcal{H}_{\text{rungs}} | \psi_{gs} \rangle|^2}{\Delta E_n}. \tag{3.23}$$

The most dominant excited states are the $3(n_l - 1)$ states

$$|\psi_n\rangle = \left\{ \begin{array}{l} |0_{y_1} 0_{y_1+1}\rangle \\ |1_{y_1} -1_{y_1+1}\rangle \\ |-1_{y_1} 1_{y_1+1}\rangle \end{array} \right\} \otimes \left(\bigotimes_{\substack{y=1 \\ y \neq y_1, y_1+1}}^{n_l} |s_y\rangle \right) \tag{3.24}$$

with $y_1 = 1, 2, \dots, n_l - 1$. The energy gap ΔE_n to these states is twice as large as in (3.2):

$$\Delta E_n \simeq \frac{\pi^2 J}{L}. \tag{3.25}$$

The numerator of (3.23) gives the factor a^2 for each of the states (3.24). Hence the second-order energy of the ground state is

$$E_{gs}^{(2)} \simeq - \frac{3(n_l - 1)a^2}{\pi^2} \left(\frac{LJ_\perp}{J} \right)^2 \frac{J}{L}. \tag{3.26}$$

Although it is complicated to calculate the second-order energy of the first excited state (3.13), we can see that the size dependence is the same as in (3.26):

$$E_{\text{ex}}^{(2)} \sim a^2 \left(\frac{LJ_{\perp}}{J} \right)^2 \frac{J}{L}. \quad (3.27)$$

We thereby have

$$\Delta E \simeq \frac{J}{L} \left[\frac{\pi^2}{2} + \alpha_1 ax + \alpha_2 (ax)^2 \right] \quad (3.28)$$

where $x \equiv LJ_{\perp}/J$, and α_1 and α_2 are constants. Since a is of the order of L^0 , the expression (3.28) is consistent with the scaling form (2.1).

It is generally difficult to calculate higher-order perturbation explicitly. However, it is possible to estimate the size dependence roughly [17]. The k th-order energy is approximately given by

$$E^{(k)} \sim \sum_{\{\psi\}} \langle \psi_{\text{gs}} | \mathcal{H}_{\text{rung}} | \psi_1 \rangle \langle \psi_2 | \mathcal{H}_{\text{rung}} | \psi_3 \rangle \cdots \langle \psi_{k-1} | \mathcal{H}_{\text{rung}} | \psi_{\text{gs}} \rangle [(\Delta E_1)(\Delta E_2) \cdots (\Delta E_{k-1})]^{-1}. \quad (3.29)$$

We may estimate the dimensionality of the operator $\mathcal{H}_{\text{rung}}$ at $LJ_{\perp} \times [S]^2 \sim J_{\perp}$, because the magnetic operator S of the antiferromagnetic Heisenberg chain has the dimensionality $[S] = L^{-\beta/\nu} = L^{-1/2}$ [2]. (This rough estimate is consistent with the estimation $a = O(L^0)$.) On the other hand, each energy denominator ΔE is of the order of J/L , because the Heisenberg chain is critical. We thereby have the rough estimate

$$E^{(k)} \sim J_{\perp}^k \left(\frac{J}{L} \right)^{-(k-1)} = \frac{J}{L} \left(\frac{LJ_{\perp}}{J} \right)^k. \quad (3.30)$$

Hence the energy gap may be given by the form

$$\Delta E \sim \frac{J}{L} \sum_k \alpha_k x^k \quad (3.31)$$

with appropriate coefficients $\{\alpha_k\}$. This is consistent with the scaling ansatz (2.1).

We have not confirmed that the above perturbational expansion is convergent. However, the numerical results in figure 1 do suggest that the series converges at least over a finite region of $x = LJ_{\perp}/J$.

4. Summary

In the present paper, we have introduced the finite-size scaling form of the energy gap of the antiferromagnetic Heisenberg ladder models:

$$\Delta E(L)/J \simeq L^{-1} \tilde{\Delta}(LJ_{\perp}/J). \quad (4.1)$$

We confirmed this scaling form numerically as well as by means of perturbation theory. On the basis of the scaling theory, we discussed the criticality of the ladder models in a unified way. The difference between the ladders with even n_l and with odd n_l was attributed to the different asymptotic behaviour of the scaling function in the limit $LJ_{\perp}/J \rightarrow \infty$.

For even n_l , the energy gap develops in the form $\Delta E \sim J_{\perp}$. The whole region of $J_{\perp} > 0$ is a disordered phase with a unique ground state. (This ground state is reduced to a set of independent singlets in the limit $J_{\perp}/J \rightarrow \infty$.) We estimated for $n_l = 2$ the critical amplitude of the energy gap and the correlation length around the critical point $J_{\perp}/J = 0$.

For odd n_l , on the other hand, the whole region of $J_{\perp} > 0$ is the critical line. The energy gap vanishes in the form $\Delta E \sim L^{-1}$ for any J_{\perp} . The critical line is controlled by the $S = 1/2$ antiferromagnetic Heisenberg chain which is obtained in the limit $J_{\perp}/J \rightarrow \infty$. The central charge of the phase is hence unity.

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